

# FUZZY ALMOST QUADRATIC FUNCTIONS

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**ABSTRACT.** We approximate a fuzzy almost quadratic function by a quadratic function in a fuzzy sense. More precisely, we establish a fuzzy Hyers–Ulam–Rassias stability of the quadratic functional equation  $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ . Our result can be regarded as a generalization of the stability phenomenon in the framework of normed spaces. We also prove a generalized version of fuzzy stability of the Pexiderized quadratic functional equation  $f(x+y) + f(x-y) = 2g(x) + 2h(y)$ .

## 1. INTRODUCTION AND PRELIMINARIES

In order to construct a fuzzy structure on a linear space, A. K. Katsaras [13] defined the notion of fuzzy norm on a linear space. Later, a few mathematicians have introduced and discussed several notions of fuzzy norm from different points of view [9, 15, 28]. In particular, T. Bag and S. K. Samanta [2], gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. They also studied some nice properties of the fuzzy norm in [3].

In mathematical analysis we may meet the following stability problem: “Assume that a function satisfies a functional equation approximately according to some convention. Is it then possible to find near this function a function satisfying the equation accurately?” In 1940, S. M. Ulam [27] posed the first stability problem. In the next year, D. H. Hyers [10] gave a partial affirmative answer to the question of Ulam. Hyers’ theorem was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The paper [22] of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. We refer the interested readers for more information on such problems to e.g. [4, 8, 11, 12, 24, 25].

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The functional equation

$$f(x+y) + f(x-y) = 2g(x) + 2h(y)$$

is said to be a Pexiderized quadratic functional equation. In the case that  $f = g = h$ , it is called the quadratic functional equation. The first author treating the stability of the quadratic equation was F. Skof [26] by proving that if  $f$  is a mapping from a normed space  $X$  into a Banach space  $Y$  satisfying  $\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon$  for some  $\epsilon > 0$ , then there is a unique quadratic function  $g : X \rightarrow Y$  such that  $\|f(x) - g(x)\| \leq \frac{\epsilon}{2}$ . P. W. Cholewa [5] extended Skof's theorem by replacing  $X$  by an abelian group  $G$ . Skof's result was later generalized by S. Czerwik [6] in the spirit of Hyers–Ulam–Rassias. S.-M. Jung and P. K. Sahoo [17], and K.W. Jun and Y. H. Lee [16] proved the stability of quadratic equation of Pexider type. The stability problem of the quadratic equation has been extensively investigated by a number of mathematicians, see [7, 8, 17, 20, 23] and references therein.

In this paper, we use the definition of a fuzzy normed space given in [2] to exhibit two reasonable fuzzy versions of stability for (Pexiderized) quadratic functional equation in the fuzzy normed linear space setting. More precisely, we approximate a function  $f$  from a space  $X$  to a fuzzy Banach space  $Y$  by a quadratic function  $Q : X \rightarrow Y$  in a fuzzy sense. In fact, we obtain a fuzzy Hyers–Ulam–Rassias stability of the quadratic equation in section 2 and a generalized version of fuzzy stability of a Pexiderized quadratic equation in section 3.

Some fuzzy stability results have been already established for the Cauchy equation  $f(x+y) = f(x) + f(y)$  in [18] and for the Jensen equation in [19].

Following [2], we give our notion of a fuzzy norm.

**Definition 1.1.** Let  $X$  be a real linear space. A function  $N : X \times \mathbb{R} \rightarrow [0, 1]$  (the so-called fuzzy subset) is said to be a fuzzy norm on  $X$  if for all  $x, y \in X$  and all  $s, t \in \mathbb{R}$ ,

- (N1)  $N(x, c) = 0$  for  $c \leq 0$ ;
- (N2)  $x = 0$  if and only if  $N(x, c) = 1$  for all  $c > 0$ ;
- (N3)  $N(cx, t) = N(x, \frac{t}{|c|})$  if  $c \neq 0$ ;
- (N4)  $N(x+y, s+t) \geq \min\{N(x, s), N(y, t)\}$ ;
- (N5)  $N(x, \cdot)$  is a non-decreasing function on  $\mathbb{R}$  and  $\lim_{t \rightarrow \infty} N(x, t) = 1$ .

The pair  $(X, N)$  is called a fuzzy normed linear space. One may regard  $N(x, t)$  as the truth value of the statement ‘the norm of  $x$  is less than or equal to the real number  $t$ ’.

**Example 1.2.** Let  $(X, \|\cdot\|)$  be a normed linear space. One can easily verify that for each  $k > 0$ ,

$$N_k(x, t) = \begin{cases} \frac{t}{t+k\|x\|} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

defines a fuzzy norm on  $X$ .

**Example 1.3.** Let  $(X, \|\cdot\|)$  be a normed linear space. Then

$$N(x, t) = \begin{cases} 0 & t \leq \|x\| \\ 1 & t > \|x\| \end{cases}$$

is a fuzzy norm on  $X$ .

Let  $(X, N)$  be a fuzzy normed linear space. Let  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  is said to be convergent if there exists  $x \in X$  such that

$$\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$$

for all  $t > 0$ . In that case,  $x$  is called the limit of the sequence  $\{x_n\}$  and we denote it by  $N - \lim x_n = x$ .

A sequence  $\{x_n\}$  in  $X$  is called Cauchy if for each  $\varepsilon > 0$  and each  $t > 0$  there exists  $n_0$  such that for all  $n \geq n_0$  and all  $p > 0$  we have  $N(x_{n+p} - x_n, t) > 1 - \varepsilon$ .

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

## 2. FUZZY HYERS–ULAM–RASSIAS STABILITY OF THE QUADRATIC EQUATION

Let  $f$  be a function from a fuzzy normed space  $(X, N)$  into a fuzzy Banach space  $(Y, N')$  and  $q \neq \frac{1}{2}$ . The function  $f$  is called a *fuzzy  $q$ -almost quadratic function*, if

$$(2.1) \quad N'(f(x+y) + f(x-y) - 2f(x) - 2f(y), t+s) \geq \min\{N(x, t^q), N(y, s^q)\}$$

for all  $x, y \in X$  and all  $s, t \in [0, \infty)$ .

The following result gives a Hyers–Ulam–Rassias stability of the quadratic equation  $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ .

**Theorem 2.1.** *Let  $q > \frac{1}{2}$  and  $f$  be a fuzzy  $q$ -almost quadratic function from a fuzzy normed space  $(X, N)$  into a fuzzy Banach space  $(Y, N')$ . Then there is a unique quadratic*

function  $Q : X \rightarrow Y$  such that for each  $x \in X$ ,

$$(2.2) \quad N'(Q(x) - f(x), t) \geq N(x, (\frac{2^{2-p} - 1}{4})^q t^q) \quad (x \in X, t > 0),$$

where  $p = \frac{1}{q}$ .

*Proof.* Put  $x = y$  and  $s = t$  in (2.1) to obtain

$$(2.3) \quad N'(f(2x) - 4f(x), 2t) \geq N(x, t^q) \quad (x \in X, t > 0).$$

Replacing  $x$  by  $2^n x$  in (2.3), we see that

$$(2.4) \quad N'(f(2^{n+1}x) - 4f(2^n x), 2t) \geq N(x, \frac{t^q}{2^n}) \quad (x \in X, n \geq 0, t > 0).$$

It follows that

$$N'(f(2^{n+1}x) - 4f(2^n x), 2^{\frac{n}{q}+1} t^{\frac{1}{q}}) \geq N(x, t) \quad (x \in X, n \geq 0, t > 0).$$

Whence

$$(2.5) \quad N'(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, t^p 2^{n(p-2)+1}) \geq N(x, t) \quad (x \in X, n \geq 0, t > 0)$$

where  $p = \frac{1}{q}$ . If  $n > m \geq 0$ , then

$$\begin{aligned} (2.6) \quad & N'(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^n t^p 2^{k(p-2)+1}) \\ & \geq N'(\sum_{k=m+1}^n (\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1} x)}{4^{k-1}}), \sum_{k=m+1}^n t^p 2^{k(p-2)+1}) \\ & \geq \min \bigcup_{k=m+1}^n \{N'(\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1} x)}{4^{k-1}}, t^p 2^{k(p-2)+1})\} \\ & \geq N(x, t) \quad (x \in X, t > 0). \end{aligned}$$

Let  $c > 0$  and  $\varepsilon$  be given. Since  $\lim_{t \rightarrow \infty} N(x, t) = 1$ , there is some  $t_0 > 0$  such that

$$N(x, t_0) \geq 1 - \varepsilon.$$

Fix some  $t > t_0$ . The convergence of the series  $\sum_{n=1}^{\infty} t^p 2^{n(p-2)+1}$  guarantees that there exists some  $n_0 \geq 0$  such that for each  $n > m \geq n_0$ , the inequality  $\sum_{k=m+1}^n t^p 2^{k(p-2)+1} < c$

holds. It follows that,

$$\begin{aligned} N'(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, c) &\geq N'(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^n t_0^p 2^{k(p-2)+1}) \\ &\geq N(x, t_0) \\ &\geq 1 - \varepsilon. \end{aligned}$$

Hence  $\{\frac{f(2^n x)}{4^n}\}$  is a Cauchy sequence in  $(Y, N')$ . Since  $(Y, N')$  is a fuzzy Banach space, this sequence converges to some  $Q(x) \in Y$ . Hence, we can define a mapping  $Q : X \rightarrow Y$ , by  $Q(x) := N' - \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n}$ . Moreover, if we put  $m = 0$  in (2.6) we observe that

$$N'(\frac{f(2^n x)}{4^n} - f(x), \sum_{k=1}^n t^p 2^{k(p-2)+1}) \geq N(x, t).$$

Therefore,

$$(2.7) \quad N'(\frac{f(2^n x)}{4^n} - f(x), t) \geq N(x, \frac{t^q}{(\sum_{k=1}^n 2^{k(p-2)+1})^q}).$$

Next we will show that  $Q$  is quadratic. Let  $x, y \in X$ , then we have

$$\begin{aligned} &N'(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \geq \\ \min &\{N'(Q(x+y) - \frac{f(2^n(x+y))}{4^n}, \frac{t}{5}), N'(Q(x-y) - \frac{f(2^n(x-y))}{4^n}, \frac{t}{5}), \\ &N'(2\frac{f(2^n x)}{4^n} - 2T(x), \frac{t}{5}), N'(2\frac{f(2^n y)}{4^n} - 2T(y), \frac{t}{5}), \\ &N'(\frac{f(2^n(x+y))}{4^n} - \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}, \frac{t}{5})\}. \end{aligned}$$

The first four terms on the right hand side of the above inequality tend to 1 as  $n \rightarrow \infty$  and the fifth term, by (2.1) is greater than or equal to

$$\min\{N(2^n x, (\frac{4^n t}{10})^q), N(2^n y, (\frac{4^n t}{10})^q)\} = \min\{N(x, 2^{(2q-1)n} (\frac{t}{10})^q), N(y, 2^{(2q-1)n} (\frac{t}{10})^q)\},$$

which tends to 1 as  $n \rightarrow \infty$ . Therefore

$$N'(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) = 1$$

for each  $x, y \in X$  and  $t > 0$ . This means that  $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$  for each  $x, y \in X$ . Next we approximate the difference between  $f$  and  $Q$  in a fuzzy sense.

For every  $x \in X$  and  $t, s > 0$ , by (2.7), for large enough  $n$ , we have

$$\begin{aligned} N'(Q(x) - f(x), t) &\geq \min\{N'(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}), N'(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2})\} \\ &\geq N(x, \frac{t^q}{(\sum_{k=1}^n 2^{k(p-2)+2})^q}) \\ &\geq N(x, (\frac{2^{2-p} - 1}{4})^q t^q). \end{aligned}$$

Let  $Q'$  be another quadratic function from  $X$  to  $Y$  which satisfies (2.2). Since for each  $n \in \mathbb{N}$ ,

$$Q(2^n x) = 4^n Q(x) \text{ \& } Q'(2^n x) = 4^n Q'(x),$$

we have

$$\begin{aligned} N'(Q(x) - Q'(x), t) &= N'(Q(2^n x) - Q'(2^n x), 4^n t) \\ &\geq \min\{N'(Q'(2^n x) - f(2^n x), \frac{4^n t}{2}), N'(f(2^n x) - Q(2^n x), \frac{4^n t}{2})\} \\ &\geq N(2^n x, (\frac{2^{2-p} - 1}{4})^q 4^{(n-\frac{1}{2})q} t^q) \\ &= N(x, (\frac{2^{2-p} - 1}{4})^q \frac{4^{nq} t^q}{2^n 2^q}) \end{aligned}$$

for each  $n \in \mathbb{N}$ . Due to  $q > \frac{1}{2}$ ,  $\lim_{n \rightarrow \infty} N(x, (\frac{2^{2-p}-1}{4})^q \frac{4^{nq} t^q}{2^n 2^q}) = 1$  for each  $x \in X$  and  $t > 0$ . Therefore  $Q = Q'$ .  $\square$

*Remark 2.2.* If  $N'(Q(x) - f(x), \cdot)$  is assumed to be right continuous at each point of  $(0, \infty)$  then we get a better fuzzy approximation than (2.2) as follows.

We have

$$\begin{aligned} N'(Q(x) - f(x), t + s) &\geq \min\{N'(Q(x) - \frac{f(2^n x)}{4^n}, s), N'(\frac{f(2^n x)}{4^n} - f(x), t)\} \\ &\geq N(x, \frac{t^q}{(\sum_{k=1}^n 2^{k(p-2)+1})^q}) \\ &\geq N(x, (\frac{2^{2-p} - 1}{2})^q t^q). \end{aligned}$$

Tending  $s$  to zero we infer

$$N'(Q(x) - f(x), t) \geq N(x, (\frac{2^{2-p} - 1}{2})^q t^q) \quad (x \in X, t > 0).$$

**Example 2.3.** Let  $X$  be a normed algebra. Using the notation of Example 1.2, let  $N = N_1$  and  $N' = N_2$ . Define  $f : (X, N) \rightarrow (X, N')$  by  $f(x) = x^2 + \|x\|x_0$ , where  $x_0$  is a unit vector in  $X$ . A straightforward computation shows that

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = (\|x+y\| + \|x-y\| - 2\|x\| - 2\|y\|)x_0$$

and

$$N(f(x+y) + f(x-y) - 2f(x) - 2f(y), s+t) \geq \min\{N'(x, t), N'(y, s)\}.$$

Therefore the conditions of Theorem 2.1 for  $q = 1$  holds. The fuzzy difference between  $Q(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = x^2$  and  $f(x)$  is equal to

$$N(f(x) - Q(x), t) = \frac{t}{t + \|x\|} = N(x, t) \geq N'(x, t) \geq N'(x, \frac{t}{2}).$$

Using Example 1.3, Theorem 2.1 can be regarded as a generalization of the classical stability result in the framework of normed spaces (see [11]).

**Theorem 2.4.** *Let  $f$  be a function from a normed space  $(X, \|\cdot\|)$  into a Banach space  $(Y, \|\cdot\|)$ . Let for some  $p > 2$ ,*

$$(2.8) \quad \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \|x\|^p + \|y\|^p$$

*for all  $x, y \in X$ . Then there is a unique quadratic additive function  $Q : X \rightarrow Y$  such that*

$$(2.9) \quad \|T(x) - f(x)\| \leq \frac{4}{2^{2-p} - 1} \|x\|^p \quad (x \in X).$$

*Remark 2.5.* Using the Hyers' type sequence  $\{4^n f(2^{-n}x)\}$  one can get 'dual' versions of Theorem 2.1 when  $q < \frac{1}{2}$ .

### 3. A FUZZY GENERAL STABILITY OF THE PEXIDERIZED QUADRATIC EQUATION

In this section, we generalize the norm version of stability of a Pexiderized quadratic equation to the framework of fuzzy normed spaces. Due to some technical reasons, we first examine the stability for odd and even functions and then we apply our results to a general function.

Throughout this section we assume that  $X$  is a linear space,  $(Y, N)$  is a fuzzy Banach space and  $(Z, N')$  is a fuzzy normed space. In addition, we suppose that  $\varphi : X \times X \rightarrow Z$  is a mapping such that

$$(3.1) \quad \varphi(2x, 2y) = \alpha \varphi(x, y)$$

for some  $\alpha \in \mathbb{R}$  and all  $x, y \in X$ .

**Proposition 3.1.** *Suppose that  $0 < |\alpha| < 2$  and that  $f, g$  and  $h$  are odd functions from  $X$  to  $Y$  such that*

$$(3.2) \quad N(f(x+y) + f(x-y) - 2g(x) - 2h(y), t) \geq N'(\varphi(x, y), t)$$

*for each  $x, y \in X$  and  $t \in \mathbb{R}$ . Then there is a unique additive mapping  $T : X \rightarrow X$  such that*

$$N(f(x) - T(x), t) \geq N''(x, \frac{2 - |\alpha|}{4}t),$$

$$(3.3) \quad N(g(x) + h(x) - T(x), t) \geq N''(x, \frac{6 - 3|\alpha|}{14 - |\alpha|}t),$$

where

$$(3.4) \quad N''(x, t) = \min\{N'(\varphi(x, x), t/3), N'(\varphi(x, 0), t/3), N'(\varphi(0, x), t/3)\}.$$

*Proof.* Noting to (N3), it is sufficient to prove the theorem in the case that  $0 < \alpha < 2$ . By changing the roles of  $x$  and  $y$  in (3.2), we get

$$(3.5) \quad N(f(x+y) - f(x-y) - 2g(y) - 2h(x), t) \geq N'(\varphi(y, x), t).$$

It follows from (3.2), (3.5) and (N4) that

$$(3.6) \quad \begin{aligned} N(f(x+y) - g(x) - h(y) - g(y) - h(x), t) &\geq \\ \min \{N'(\varphi(x, y), t), N'(\varphi(y, x), t)\}. \end{aligned}$$

If we put  $y = 0$  in (3.6), we obtain

$$(3.7) \quad N(f(x) - g(x) - h(x), t) \geq \min\{N'(\varphi(x, 0), t), N'(\varphi(0, x), t)\}.$$

It follows from (3.6), (3.7) and (N4)

$$(3.8) \quad \begin{aligned} N(f(x+y) - f(x) - f(y), 3t) &\geq \min \{N'(\varphi(x, y), t), N'(\varphi(y, x), t), \\ N'(\varphi(x, 0), t), N'(\varphi(0, x), t), \\ N'(\varphi(y, 0), t), N'(\varphi(0, y), t)\}. \end{aligned}$$

Then  $N''(2^n x, t) = N''(x, \frac{t}{\alpha^n})$ . If we put  $x = y$  in (3.8), we see that

$$(3.9) \quad N(f(2x) - 2f(x), t) \geq N''(x, t).$$



Replacing  $x$  by  $2^n x$  in (3.9) we have

$$\begin{aligned} N\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, t\right) &= N(f(2^{n+1}x) - f(2^n x), 2^n t) \\ &\geq N''(2^n x, 2^n t) \\ &\geq N''\left(x, \left(\frac{2}{\alpha}\right)^n t\right), \end{aligned}$$

whence

$$N\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \left(\frac{\alpha}{2}\right)^n t\right) \geq N''(x, t).$$

Therefore for each  $n > m \geq 0$ ,

$$\begin{aligned} (3.10) \quad N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \sum_{k=m+1}^n \left(\frac{\alpha}{2}\right)^{k-1} t\right) &= N\left(\sum_{k=m+1}^n \frac{f(2^k x)}{2^k} - \frac{f(2^{k-1} x)}{2^{k-1}}, \sum_{k=m+1}^n \left(\frac{\alpha}{2}\right)^{k-1} t\right) \\ &\geq \min \bigcup_{k=m+1}^n \left\{ N\left(\frac{f(2^k x)}{2^k} - \frac{f(2^{k-1} x)}{2^{k-1}}, \left(\frac{\alpha}{2}\right)^{k-1} t\right) \right\} \\ &\geq N''(x, t). \end{aligned}$$

Let  $t_0 > 0$  and  $\varepsilon > 0$  be given. Thanks to the fact that  $\lim_{s \rightarrow \infty} N''(x, s) = 1$ , we can find some  $t_1 > t_0$  such that

$$N''(x, t_1) > 1 - \varepsilon.$$

By the convergence of the series  $\sum_{n=1}^{\infty} \left(\frac{\alpha}{2}\right)^n t_1$  we can find some  $n_0 \in \mathbb{N}$  such that for each  $n > m \geq n_0$ ,

$$\sum_{k=m+1}^n \left(\frac{\alpha}{2}\right)^{k-1} t_1 < t_0.$$

Therefore

$$\begin{aligned} N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, t_0\right) &\geq N\left(\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}, \sum_{k=m+1}^n \left(\frac{\alpha}{2}\right)^{k-1} t_1\right) \\ &\geq N''(x, t_1) \\ &> 1 - \varepsilon. \end{aligned}$$

Hence,  $\left\{\frac{f(2^n x)}{2^n}\right\}$  is a Cauchy sequence in  $(Y, N)$ . Since  $(Y, N)$  is a Banach fuzzy space, this sequence converges to some point  $T(x) \in Y$ . Define  $T : X \rightarrow Y$  by  $T(x) := N - \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ .

Fix  $x, y \in X$  and  $t > 0$ . It follows from (3.8) that

$$\begin{aligned}
 (3.11) \quad N\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right) &= N\left(f(2^n(x+y)) - f(2^n x) - f(2^n y), \frac{2^n t}{4}\right) \\
 &\geq \min\left\{N'(\varphi(x, y), \frac{2^n t}{12\alpha^n}), N'(\varphi(y, x), \frac{2^n t}{12\alpha^n}), \right. \\
 &\quad N'(\varphi(x, 0), \frac{2^n t}{12\alpha^n}), N'(\varphi(0, x), \frac{2^n t}{12\alpha^n}), \\
 &\quad \left. N'(\varphi(y, 0), \frac{2^n t}{12\alpha^n}), N'(\varphi(0, y), \frac{2^n t}{12\alpha^n})\right\}
 \end{aligned}$$

for all  $n$ . Moreover,

$$\begin{aligned}
 (3.12) \quad N(T(x+y) - T(x) - T(y), t) &\geq \min\left\{N\left(T(x+y) - \frac{f(2^n(x+y))}{2^n}, \frac{t}{4}\right), \right. \\
 &\quad N\left(T(x) - \frac{f(2^n x)}{2^n}, \frac{t}{4}\right), N\left(T(y) - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right), \\
 &\quad \left. N\left(\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}, \frac{t}{4}\right)\right\}
 \end{aligned}$$

for all  $n$ . Since each factor in the right hand side of (3.11) and (3.12) tends to 1 as  $n \rightarrow \infty$ , one can easily see that  $N(T(x+y) - T(x) - T(y), t) = 1$  whence

$$T(x+y) = T(x) + T(y).$$

Furthermore, using (3.10) with  $m = 0$ , we see that for large  $n$ ,

$$\begin{aligned}
 (3.13) \quad N(T(x) - f(x), t) &\geq \min\left\{N\left(T(x) - \frac{f(2^n x)}{2^n}, \frac{t}{2}\right), N\left(\frac{f(2^n x)}{2^n} - f(x), \frac{t}{2}\right)\right\} \\
 &\geq \min\left\{N\left(T(x) - \frac{f(2^n x)}{2^n}, \frac{t}{2}\right), N''\left(x, \frac{t}{2 \sum_{k=1}^n \left(\frac{\alpha}{2}\right)^{k-1}}\right)\right\} \\
 &\geq N''\left(x, \frac{t}{2 \sum_{k=0}^{\infty} \left(\frac{\alpha}{2}\right)^{k-1}}\right) \\
 &= N''\left(x, \frac{2-\alpha}{4}t\right).
 \end{aligned}$$

It follows from (3.7) and (3.13) that

$$\begin{aligned}
 N(g(x) + h(x) - T(x), \frac{14-\alpha}{12}t) &, \\
 &\geq \min\left\{N(f(x) - T(x), t), N(g(x) + h(x) - f(x), \frac{2-\alpha}{12}t)\right\} \\
 &\geq \min\left\{N''\left(x, \frac{2-\alpha}{4}t\right), N'(\varphi(x, 0), \frac{2-\alpha}{12}t), N'(\varphi(0, x), \frac{2-\alpha}{12}t)\right\} \\
 &\geq N''\left(x, \frac{2-\alpha}{4}t\right),
 \end{aligned}$$

whence we obtained (3.3).

The proof for the uniqueness assertion is similar to Theorem 2.1.  $\square$

**Proposition 3.2.** *Suppose that  $0 < |\alpha| < 4$ , and that  $f, g$  and  $h$  are even functions from  $X$  to  $Y$  such that  $f(0) = g(0) = h(0) = 0$  and*

$$(3.14) \quad N(f(x+y) + f(x-y) - 2g(x) - 2h(y), t) \geq N'(\varphi(x, y), t)$$

*for each  $x, y \in X$  and  $t \in \mathbb{R}$ . Then there is a unique quadratic mapping  $Q : X \rightarrow Y$  such that*

$$N(Q(x) - f(x), t) \geq N''(x, \frac{4-|\alpha|}{16}t),$$

$$N(Q(x) - g(x), t) \geq N''(x, \frac{12-3|\alpha|}{52-|\alpha|}t),$$

$$N(Q(x) - h(x), t) \geq N''(x, \frac{12-3|\alpha|}{52-|\alpha|}t),$$

where  $N''(x, t)$  is defined by (3.4).

*Proof.* Noting to (N3), it is sufficient to prove the theorem in the case that  $0 < \alpha < 4$ . Change the roles of  $x$  and  $y$  in (3.14) to get

$$(3.15) \quad N(f(x+y) + f(x-y) - 2g(y) - 2h(x), t) \geq N'(\varphi(y, x), t).$$

Put  $y = x$  in (3.14) to obtain

$$(3.16) \quad N(f(2x) - 2g(x) - 2h(x), t) \geq N'(\varphi(x, x), t).$$

Put  $x = 0$  in (3.14) to obtain

$$(3.17) \quad N(2f(y) - 2h(y), t) \geq N'(\varphi(0, y), t).$$

Similarly, putting  $y = 0$  in (3.14) we get

$$(3.18) \quad N(2f(x) - 2g(x), t) \geq N'(\varphi(x, 0), t).$$

Combining (3.15), (3.17), (3.18) we get

$$(3.19) \quad \begin{aligned} N(f(x+y) - f(x-y) - 2f(x) - 2f(y), t) &\geq \min\{N'(\varphi(x, y), t/3), \\ &N'(\varphi(x, 0), t/3), N'(\varphi(0, y), t/3)\}. \end{aligned}$$

Setting  $y = x$  in (3.19) we have

$$(3.20) \quad N(f(2x) - 4f(x), t) \geq N''(x, t),$$

where  $N''(x, t)$  is defined by

$$N''(x, t) = \min\{N'(\varphi(x, x), t/3), N'(\varphi(x, 0), t/3), N'(\varphi(0, x), t/3)\}.$$

By (3.1),

$$(3.21) \quad N''(2^n x, t) = N''(x, \frac{t}{\alpha^n}),$$

for each  $n \geq 0$  and  $x \in X$ . It follows from (3.20) and (3.21) that

$$(3.22) \quad N(f(2^{n+1}x) - 4f(2^n x), t) \geq N''(x, \frac{t}{\alpha^n}).$$

By (3.22),

$$\begin{aligned} N(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, t) &= N(f(2^{n+1}x) - 4f(2^n x), 4^{n+1}t) \\ &\geq N''(x, \frac{4^{n+1}t}{\alpha^n}). \end{aligned}$$

or equivalently,

$$N(\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}, \frac{\alpha^n t}{4^{n+1}}) \geq N''(x, t).$$

Therefore for each  $n > m \geq 0$ ,

$$\begin{aligned} N(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^n \frac{\alpha^{k-1} t}{4^k}) &= N(\sum_{k=m+1}^n (\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1} x)}{4^{k-1}}), \sum_{k=m+1}^n \frac{\alpha^{k-1} t}{4^k}) \\ (3.23) \quad &\geq \min \bigcup_{k=m+1}^n \{N(\frac{f(2^k x)}{4^k} - \frac{f(2^{k-1} x)}{4^{k-1}}, \frac{\alpha^{k-1} t}{4^k})\} \\ &\geq N''(x, t). \end{aligned}$$

Let  $t_0 > 0$ ,  $\varepsilon > 0$  be given. Since  $\lim_{t \rightarrow \infty} N''(x, t) = 1$  there is some  $t_1 > t_0$  such that  $N''(x, t_1) > 1 - \varepsilon$ . The convergence of the series  $\sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{4^k} t_1$  gives some  $n_0$  such that  $\sum_{k=m+1}^n \frac{\alpha^{k-1}}{4^k} t_1 < t_0$  for each  $n > m \geq n_0$ . It follows that for each  $n > m > n_0$ ,

$$\begin{aligned} N(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, t_0) &\geq N(\frac{f(2^n x)}{4^n} - \frac{f(2^m x)}{4^m}, \sum_{k=m+1}^n \frac{\alpha^{k-1}}{4^k} t_1) \\ &\geq N''(x, t_0) \\ &> 1 - \varepsilon. \end{aligned}$$

This shows that  $\{\frac{f(2^n x)}{4^n}\}$  is Cauchy sequence in the fuzzy Banach space  $(Y, N)$ , therefore it is convergence to some  $Q(x)$ . So we can define a mapping  $Q : X \rightarrow Y$  by  $Q(x) = N - \lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n}$ .

Fix  $x, y \in X$  and  $t > 0$ . It follows from (3.19) that

$$\begin{aligned}
 (3.24) \quad N\left(\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}\right) \\
 = N(f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y), \frac{4^n t}{5}) \\
 \geq \min\{N'(\varphi(x, y), \frac{4^n t}{15\alpha^n}), \\
 N'(\varphi(x, 0), \frac{4^n t}{15\alpha^n}), N'(\varphi(0, y), \frac{4^n t}{15\alpha^n})\}
 \end{aligned}$$

for all  $n$ . Moreover,

$$\begin{aligned}
 (3.25) \quad N(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) \geq \min\{N(Q(x+y) - \frac{f(2^n(x+y))}{4^n}, \frac{t}{5}), \\
 N(Q(x-y) - \frac{f(2^n(x-y))}{4^n}, \frac{t}{5}), N(2Q(x) - 2\frac{f(2^n x)}{4^n}, \frac{t}{5}), \\
 N(2Q(y) - 2\frac{f(2^n y)}{4^n}, \frac{t}{5}), \\
 N(\frac{f(2^n(x+y))}{4^n} + \frac{f(2^n(x-y))}{4^n} - 2\frac{f(2^n x)}{4^n} - 2\frac{f(2^n y)}{4^n}, \frac{t}{5})\}
 \end{aligned}$$

for all  $n$ . Since each factor in the right hand side of (3.24) and (3.25) tends to 1 as  $n \rightarrow \infty$ , one can easily see that  $N(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t) = 1$  whence

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y).$$

Furthermore, using (3.23) with  $m = 0$ , we see that for large  $n$ ,

$$\begin{aligned}
 (3.26) \quad N(Q(x) - f(x), t) &\geq \min\{N(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}), N(\frac{f(2^n x)}{4^n} - f(x), \frac{t}{2})\} \\
 &\geq \min\{N(Q(x) - \frac{f(2^n x)}{4^n}, \frac{t}{2}), N''(x, \frac{4t}{\sum_{k=1}^n (\frac{\alpha}{4})^{k-1}})\} \\
 &\geq N''(x, \frac{4t}{\sum_{k=0}^{\infty} (\frac{\alpha}{4})^k}) \\
 (3.27) \quad &= N''(x, \frac{4-\alpha}{16}t).
 \end{aligned}$$

It follows from (3.18) and (3.26) that

$$\begin{aligned}
N(Q(x) - g(x), \frac{52 - \alpha}{48}t) &\geq \min\{N(Q(x) - f(x), t), N(f(x) - g(x), \frac{4 - \alpha}{48}t)\} \\
&\geq \min\{N''(x, \frac{4 - \alpha}{16}t), N'(\varphi(x, 0), \frac{4 - \alpha}{48}t)\} \\
&\geq N''(x, \frac{4 - \alpha}{16}t)
\end{aligned}$$

whence

$$N(Q(x) - g(x), t) \geq N''(x, \frac{12 - 3\alpha}{52 - \alpha}t).$$

A similar inequality holds for  $h$ . The uniqueness assertion can be proved by a known strategy as in Theorem 2.1.  $\square$

**Theorem 3.3.** *Let  $|\alpha| < 2$ , let  $f$  be a mapping from  $X$  to  $Y$  such that  $f(0) = 0$  and*

$$(3.28) \quad N(f(x + y) + f(x - y) - 2f(x) - 2f(y), t) \geq N'(\varphi(x, y), t)$$

*for all  $x, y \in X$  and all  $t > 0$ . Then there are unique mappings  $T$  and  $Q$  from  $X$  to  $Y$  such that  $T$  is additive,  $Q$  is quadratic and*

$$N(f(x) - T(x) - Q(x), t) \geq M(x, \min\{\frac{2 - \alpha}{8}, \frac{4 - \alpha}{32}\}t) \quad (x \in X, t > 0),$$

*where*

$$\begin{aligned}
M(x, t) &= \min\{N'(\varphi(x, x), t/3), N'(\varphi(-x, -x), t/3), \\
&\quad N'(\varphi(x, 0), t/3), N'(\varphi(0, x), t/3), \\
&\quad N'(\varphi(-x, 0), t/3), N'(\varphi(0, -x), t/3)\}.
\end{aligned}$$

*Proof.* Passing to the odd part  $f^o$  and even part  $f^e$  of  $f$  we deduce from (3.28) that

$$\begin{aligned}
N(f^o(x + y) + f^o(x - y) - 2f^o(x) - 2f^o(y), t) \\
\geq \min\{N'(\varphi(x, y), t), N'(\varphi(-x, -y), t)\}
\end{aligned}$$

and

$$\begin{aligned}
N(f^e(x + y) + f^e(x - y) - 2f^e(x) - 2f^e(y), t) \\
\geq \min\{N'(\varphi(x, y), t), N'(\varphi(-x, -y), t)\}.
\end{aligned}$$

Using the proofs of Propositions 3.1 and 3.2 we get unique additive mapping  $T$  and unique quadratic mapping  $Q$  satisfying

$$N(f^o(x) - T(x), t) \geq M(x, \frac{2 - |\alpha|}{4}t),$$

and

$$N(f^e(x) - Q(x), t) \geq M(x, \frac{4 - |\alpha|}{16}t).$$

Therefore

$$\begin{aligned} N(f(x) - T(x) - Q(x), t) &\geq \min\{N(f^o - T(x), \frac{t}{2}), N(f^e - Q(x), \frac{t}{2})\} \\ &\geq \min\{M(x, \frac{2 - |\alpha|}{8}t), M(x, \frac{4 - |\alpha|}{32}t)\} \\ &= M(x, \min\{\frac{2 - \alpha}{8}, \frac{4 - \alpha}{32}\}t) \quad (x \in X, t > 0). \end{aligned}$$

□

The following example provides an illustration.

**Example 3.4.** Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space,  $Y$  be a normed space and  $Z$  be the real line  $\mathbb{R}$ . Let  $N$  and  $N'$  be the fuzzy norms on  $Y$  and  $\mathbb{R}$ , defined by Example 1.2 with  $k = 1$ , respectively. Suppose that the fuzzy metric  $N$ , makes  $Y$  into a fuzzy Banach space. Fix elements  $x_0, y_0$  and  $z_0$  in  $Y$  and  $a$  in  $X$  and define

$$f(x) = \langle x, a \rangle x_0 + \|x\|^2 y_0 + \sqrt{\|x\|} z_0,$$

$$g(x) = \langle x, a \rangle x_0 + \|x\|^2 y_0,$$

$$h(x) = \|x\|^2 y_0 + \sqrt{\|x\|} z_0,$$

$$\varphi(x, y) = (\sqrt{\|x + y\|} + \sqrt{\|x - y\|} - 2\sqrt{\|y\|}) \|z_0\|$$

for each  $x, y \in X$ . One can easily verified that

$$f(x + y) + f(x - y) - 2g(x) - 2h(y) = (\sqrt{\|x + y\|} + \sqrt{\|x - y\|} - 2\sqrt{\|y\|}) z_0$$

for each  $x, y \in X$ . Therefore

$$N(f(x + y) + f(x - y) - 2g(x) - 2h(y), t) = N'(\varphi(x, y), t)$$

for each  $x, y \in X$  and  $t \in \mathbb{R}$ . Moreover,  $\varphi(2x, 2y) = \sqrt{2}\varphi(x, y)$  for each  $x, y \in X$ .

Therefore the conditions of Propositions 3.1 and 3.2 for  $f, g, h$  and  $|\alpha| = \sqrt{2} < 2$  are satisfied. It follows that odd and even parts of  $f$  can be approximated by linear and quadratic functions, respectively. In fact  $f^o$ , the odd part of  $f$ , is equal to  $f^o(x) = \langle x, a \rangle$

$x_0$  is linear and the even part of  $f$ ,  $f^e$ , is equal to  $f^e(x) = \|x\|^2 y_0 + \sqrt{\|x\|} z_0$  contains a quadratic  $Q(x) = \|x\|^2 y_0$  and

$$N(f^e(x) - Q(x), t) = N'(\sqrt{\|x\|} \|z_0\|, t) \geq N''(x, \frac{4 - \sqrt{2}}{16} t).$$

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